SOLUTIONS TO $3.5.44, 3.5.45, 3.5.46$

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Problem 3.5.44

Problem: Show that the equation of the tangent line to the ellipse:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) is

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$$

Solution:

Slope:

$$\begin{aligned} \frac{2x}{a^2} + \frac{2yy'}{b^2} =& 0 \\ y'\left(\frac{2y}{b^2}\right) =& -\frac{2x}{a^2} \\ y' =& -\frac{b^2}{a^2}\frac{2x}{2y} \\ y' =& -\frac{b^2}{a^2}\frac{x}{y} \end{aligned}$$

Equation:
At (x_0, y_0), the slope is $-\frac{b^2}{a^2}\frac{x_0}{y_0}$, so the equation of the tangent line at (x_0, y_0) is:

$$y - y_0 = \left(-\frac{b^2}{a^2}\frac{x_0}{y_0}\right)(x - x_0)$$

Simplification:
First of all, by multiplying both sides by $a^2 y_0$, we get:

$$(y - y_0)(a^2 y_0) = -b^2 x_0 (x - x_0)$$

Expanding out, we get:

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$$ya^2y_0 - a^2(y_0)^2 = -b^2x_0x + b^2(x_0)^2$$

Now rearranging, we have:

$$ya^2y_0 + b^2x_0x = a^2(y_0)^2 + b^2(x_0)^2$$

Now dividing both sides by a^2, we get:

$$yy_0 + \frac{b^2}{a^2}x_0x = (y_0)^2 + \frac{b^2}{a^2}(x_0)^2$$

And dividing both sides by b^2, we get:

$$\frac{yy_0}{b^2} + \frac{x_0x}{a^2} = \frac{(y_0)^2}{b^2} + \frac{(x_0)^2}{a^2}$$

But now, since (x_0, y_0) is on the ellipse, $\frac{(y_0)^2}{b^2} + \frac{(x_0)^2}{a^2} = 1$, we get:

$$\frac{yy_0}{b^2} + \frac{x_0x}{a^2} = 1$$

Whence,

$$\frac{x_0x}{a^2} + \frac{y_0y}{b^2} = 1$$

Problem 3.5.45

Problem: Show that the equation of the tangent line to the hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

at the point (x_0, y_0) is

$$\frac{x_0x}{a^2} - \frac{y_0y}{b^2} = 1$$

Solution:

<u>Slope:</u>

$$\begin{aligned}\frac{2x}{a^2}-\frac{2yy'}{b^2} &=0\\ y'\left(-\frac{2y}{b^2}\right) &=-\frac{2x}{a^2}\\ y' &=\frac{b^2}{a^2}\frac{2x}{2y}\\ y' &=\frac{b^2}{a^2}\frac{x}{y}\end{aligned}$$

<u>Equation:</u>
At (x_0, y_0), the slope is $\frac{b^2}{a^2}\frac{x_0}{y_0}$, so the equation of the tangent line at (x_0, y_0) is:

$$y - y_0 = \left(\frac{b^2}{a^2}\frac{x_0}{y_0}\right)(x - x_0)$$

<u>Simplification:</u>
First of all, by multiplying both sides by $a^2 y_0$, we get:

$$(y - y_0)(a^2 y_0) = b^2 x_0 (x - x_0)$$

Expanding out, we get:

$$ya^2 y_0 - a^2 (y_0)^2 = b^2 x_0 x - b^2 (x_0)^2$$

Now rearranging, we have:

$$ya^2 y_0 - b^2 x_0 x = a^2 (y_0)^2 - b^2 (x_0)^2$$

Now dividing both sides by a^2, we get:

$$yy_0 - \frac{b^2}{a^2} x_0 x = (y_0)^2 - \frac{b^2}{a^2}(x_0)^2$$

And dividing both sides by b^2, we get:

$$\frac{yy_0}{b^2} - \frac{x_0 x}{a^2} = \frac{(y_0)^2}{b^2} - \frac{(x_0)^2}{a^2}$$

But now, since (x_0, y_0) is on the hyperbola, $\frac{(x_0)^2}{a^2} - \frac{(y_0)^2}{b^2} = 1$, so $\frac{(y_0)^2}{b^2} - \frac{(x_0)^2}{a^2} = -1$, and we get:

$$\frac{yy_0}{b^2} - \frac{x_0 x}{a^2} = -1$$

Whence,

$$\frac{x_0 x}{a^2} - \frac{y_0 y}{b^2} = 1$$

Problem 3.5.46

Problem: Show that the sum of the $x-$ and $y-$ intercepts of any tangent line to the curve $\sqrt{x}+\sqrt{y}=\sqrt{c}$ is equal to c.

Solution:

Slope:

$$\begin{aligned}\frac{1}{2\sqrt{x}}+y'\left(\frac{1}{2\sqrt{y}}\right)=&0\\ y'\left(\frac{1}{2\sqrt{y}}\right)=&-\frac{1}{2\sqrt{x}}\\ y'=&-\frac{\frac{1}{2\sqrt{x}}}{\frac{1}{2\sqrt{y}}}\\ y'=&-\frac{2\sqrt{y}}{2\sqrt{x}}\\ y'=&-\frac{\sqrt{y}}{\sqrt{x}}\end{aligned}$$

Equation: At (x_0,y_0), the slope is $-\frac{\sqrt{y_0}}{\sqrt{x_0}}$, and so the equation of the tangent line at (x_0,y_0) is:

$$y-y_0=-\frac{\sqrt{y_0}}{\sqrt{x_0}}(x-x_0)$$

$y-$intercept:
To find the $y-$intercept, set $x=0$ and solve for y:

$$\begin{aligned}y-y_0=&-\frac{\sqrt{y_0}}{\sqrt{x_0}}(0-x_0)\\ y-y_0=&-\frac{\sqrt{y_0}}{\sqrt{x_0}}(-x_0)\\ y-y_0=&\sqrt{y_0}\sqrt{x_0}\\ y=&y_0+\sqrt{y_0}\sqrt{x_0}\end{aligned}$$

$x-$intercept:
To find the $x-$intercept, set $y=0$ and solve for x:

$$\begin{aligned}
0 - y_0 &= -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0) \\
-y_0 &= -\frac{\sqrt{y_0}}{\sqrt{x_0}}(x - x_0) \\
x - x_0 &= -\frac{\sqrt{x_0}}{\sqrt{y_0}}(-y_0) \\
x &= x_0 + \sqrt{x_0}\sqrt{y_0}
\end{aligned}$$

<u>Sum:</u>
The sum of the $y-$ and $x-$ intercepts is:

$$(y_0 + \sqrt{y_0}\sqrt{x_0}) + (x_0 + \sqrt{x_0}\sqrt{y_0}) = x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0$$

But the trick is that:

$$x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = (\sqrt{x_0})^2 + 2\sqrt{x_0}\sqrt{y_0} + (\sqrt{y_0})^2 = (\sqrt{x_0} + \sqrt{y_0})^2$$

But since (x_0, y_0) is on the curve $\sqrt{x} + \sqrt{y} = \sqrt{c}$, we get $\sqrt{x_0} + \sqrt{y_0} = \sqrt{c}$.

And so, finally we get that the sum of the $x-$ and $y-$ intercepts is:

$$x_0 + 2\sqrt{x_0}\sqrt{y_0} + y_0 = (\sqrt{x_0} + \sqrt{y_0})^2 = (\sqrt{c})^2 = c$$